

## A Spectral Characterization of Stochastic Matrices\*

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### 1. INTRODUCTION

Let  $F$  be a field with identity element 1, and  $A$  an  $n \times n$  matrix over  $F$ . If  $A$  is either row or column stochastic (that is, all row sums or all column sums equal 1), then 1 is a characteristic root of  $A$ . Moreover, whenever  $A$  is row or column stochastic so is  $PA$  where  $P$  is a  $n \times n$  permutation matrix; thus 1 is a characteristic root of  $PA$  for every  $n \times n$  permutation matrix  $P$ . It is shown here that this property actually characterizes stochastic matrices over  $F$  (Theorem 1). Matrices  $A$  over  $F$  for which the spectrum of  $A$  is the same as the spectrum of  $PA$  for every  $n \times n$  permutation matrix  $P$  are also characterized (Theorem 2).

### 2. MAIN RESULT

The vector space of  $n$ -tuples over  $F$ , written as columns, is denoted by  $F^n$ . If  $y \in F^n$ , then  $y^i$  denotes the  $i$ th coordinate of  $y$  ( $i = 1, 2, \dots, n$ ).

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**THEOREM 1.** *Let  $A$  be an  $n \times n$  matrix over a field  $F$  with identity element 1. If 1 is a characteristic root of  $PA$  for all  $n \times n$  permutation matrices  $P$ , then  $A$  is either row stochastic or column stochastic.*

*Proof.* If  $P$  is any  $n \times n$  permutation matrix, then there exists a nonzero vector  $x_P \in F^n$  such that  $PAx_P = x_P$ . For  $P$  fixed let

$$X_P = \{x : x \in F^n, PAx = x\}.$$

Each vector space  $X_P$  has dimension at least one. If  $I$  denotes the  $n \times n$  identity matrix over  $F$ , then we may assume without loss of generality that  $X_I$  has the least dimension of all the  $X_P$  (replace  $A$  by a suitable  $QA$ ,  $Q$  an  $n \times n$  permutation matrix).

Let  $x_I$  be a nonzero vector in  $X_I$ . If  $x_I^1 = x_I^2 = \cdots = x_I^n$ , then  $A$  is row stochastic. Hence we may assume that not all the coordinates of  $x_I$  are equal. After possibly replacing  $A$  by a suitable  $RAR^T$ ,  $R$  a permutation matrix, there exists an integer  $k$  with  $1 \leq k < n$  such that

$$x_I^i \neq x_I^j, \quad 1 \leq i \leq k < j \leq n.$$

(The  $X_I$  corresponding to  $RAR^T$  will still have minimal dimension.) If  $k > 1$ , then for  $2 \leq i \leq k$  let  $P_i$  be the  $n \times n$  permutation matrix associated with the transposition  $(i, n)$ . For  $k+1 \leq j \leq n$ , let  $Q_j$  be the  $n \times n$  permutation matrix corresponding to the transposition  $(1, j)$ .

For  $i = 2, \dots, k$  pick  $x_i \in X_{P_i}$  with  $x_i^i \neq x_i^n$ . This is possible, for otherwise  $X_{P_i}$  is a proper subspace of  $X_I$  which contradicts the minimality of the dimension of  $X_I$ . For  $j = k+1, \dots, n$  pick  $y_j \in X_{Q_j}$  with  $y_j^1 \neq y_j^j$ . Assume

$$x_I b + \sum_{i=2}^k x_i c_i + \sum_{j=k+1}^n y_j d_j = 0, \quad (1)$$

where the first summation is vacuous if  $k = 1$ . Apply  $A$  to Eq. (1) to obtain, because of  $P_i^{-1} = P_i$ ,  $Q_j^{-1} = Q_j$ ,

$$x_I b + \sum_{i=2}^k P_i x_i c_i + \sum_{j=k+1}^n Q_j y_j d_j = 0. \quad (2)$$

Subtract Eq. (2) from Eq. (1) to get

$$\sum_{i=2}^k (x_i - P_i x_i) c_i + \sum_{j=k+1}^n (y_j - Q_j y_j) d_j = 0. \quad (3)$$

Compare the  $\kappa$ th component on both sides,  $2 \leq \kappa \leq k$ . Since  $P_i$  ( $i \neq \kappa$ ) and  $Q_j$  do not change the  $\kappa$ th coordinate of their corresponding vectors, (3) implies

$$(x_\kappa^\kappa - x_\kappa^n)c_\kappa = 0$$

and therefore

$$c_\kappa = 0, \quad \kappa = 2, \dots, k. \quad (4)$$

If  $k+1 \leq \lambda \leq n-1$ , then  $P_i$  and  $Q_j$  ( $j \neq \lambda$ ) do not change the  $\lambda$ th coordinate of their corresponding vectors. Therefore

$$d_\lambda = 0, \quad \lambda = k+1, \dots, n-1. \quad (5)$$

The use of (4) and (5) in (3) gives

$$(y_n^n - y_n^1)d_n = 0$$

and hence

$$d_n = 0. \quad (6)$$

Now (4), (5), and (6) when used in Eq. (1) give  $b = 0$ . Hence the vectors  $x_1, x_2, \dots, x_k, y_{k+1}, \dots, y_n$  form a set of  $n$  linearly independent vectors.

Simplifying the notation we may state that there are  $n$  permutation matrices  $R_1, R_2, \dots, R_n$  and  $n$  linearly independent vectors  $z_1, z_2, \dots, z_n$  such that

$$R_i A z_i = z_i, \quad i = 1, 2, \dots, n.$$

Let  $e = (1, 1, \dots, 1)$ . Then

$$e R_i A z_i = e z_i, \quad i = 1, 2, \dots, n$$

and therefore

$$e(A - I)z_i = 0, \quad i = 1, 2, \dots, n.$$

Since  $z_1, z_2, \dots, z_n$  are linearly independent,

$$e(A - I) = 0$$

or

$$eA = e.$$

That is,  $A$  is column stochastic. This completes the proof of the theorem.

*Remark 1.* The proof shows a little more than the theorem states:

(i) It is sufficient to assume that  $F$  is a division ring if the statement "1 is a characteristic root of  $PA$ " is interpreted to mean that there exists  $0 \neq x \in F^n$  with  $PAx = x$ .

(ii) If  $A$  is an  $n \times n$  row stochastic matrix over  $F$  and a vector space  $X_p$  of smallest dimension contains a vector with two distinct coordinates, then  $A$  is doubly stochastic (both row and column stochastic).

*Remark 2.* It seems difficult to find sets of less than  $n!$  permutation matrices which can be used to characterize the stochastic  $n \times n$  matrices in the manner of Theorem 1. For instance, in case  $n = 4$  neither (i) all transpositions nor (ii) all even permutations nor (iii) all powers of a cyclic permutation will do. Counterexamples are given by

$$(i, ii) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad (iii) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The following corollary is an immediate consequence of Theorem 1.

**COROLLARY 1.** Let  $F_1$  be an extension field of  $F$  and  $0 \neq \alpha \in F_1$ . Suppose  $A$  is an  $n \times n$  matrix over  $F$  with  $\alpha$  a characteristic root of  $PA$  for all  $n \times n$  permutation matrices  $P$ . Then actually  $\alpha \in F$  and either all row sums of  $A$  equal  $\alpha$  or all column sums of  $A$  equal  $\alpha$ .

**COROLLARY 2.** Let  $A$  be an  $n \times n$  matrix over the field  $F$  and assume the characteristic of  $F$  is either 0 or relatively prime to  $n$ . Then two distinct nonzero elements of  $F$  cannot both be characteristic roots of  $PA$  for all  $n \times n$  permutation matrices  $P$ .

*Proof.* Suppose  $\alpha \neq 0$  and  $\beta \neq 0$ ,  $\alpha \neq \beta$ , are elements of  $F$  both of which are characteristic roots of  $PA$  for all  $n \times n$  permutation matrices  $P$ . Then by Corollary 1 all row sums of  $A$  equal  $\alpha$  and all column sums of  $A$  equal  $\beta$  (or vice versa). The sum of the entries of  $A$  via the row sums

is  $n\alpha$ , while the sum of the entries of  $A$  via the column sums is  $n\beta$ . Hence  $n\alpha = n\beta$  and the assumptions on the characteristic of  $F$  imply  $\alpha = \beta$ , a contradiction.

### 3. INVARIANCE OF THE SPECTRUM

We vary the problem by imposing a stronger condition on the matrix  $A$ . Let  $A$  be an  $n \times n$  matrix over an algebraically closed field  $F$ . If  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the  $n$  characteristic roots of  $A$ , then the spectrum of  $A$  is denoted by  $\text{sp } A = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ . Our question is: When is the spectrum of  $A$  the same (counting multiplicities) as the spectrum of  $PA$  for all  $n \times n$  permutation matrices  $P$ ? Obviously sufficient is the condition that  $A$  or  $A^T$  (the transpose of  $A$ ) has the form

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \\ & & \ddots & \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \quad (x_i \in F). \quad (7)$$

The following theorem gives the complete answer.

**THEOREM 2.** *Let  $A$  be an  $n \times n$  matrix ( $n \geq 2$ ) over an algebraically closed field  $F$ .*

(i) *If the characteristic of  $F$  is either 0 or relatively prime to  $n$ , then  $\text{sp } A = \text{sp } PA$  for all  $n \times n$  permutation matrices  $P$  if and only if  $A$  or  $A^T$  has the form given in (7).*

(ii) *If the characteristic of  $F$  divides  $n$ , then  $\text{sp } A = \text{sp } PA$  for all  $n \times n$  permutation matrices  $P$  if and only if  $A$  has the form*

$$\begin{bmatrix} \alpha_1 + \beta_1 & \alpha_1 + \beta_2 & \cdots & \alpha_1 + \beta_n \\ \alpha_2 + \beta_1 & \alpha_2 + \beta_2 & \cdots & \alpha_2 + \beta_n \\ & & \ddots & \\ \alpha_n + \beta_1 & \alpha_n + \beta_2 & \cdots & \alpha_n + \beta_n \end{bmatrix} \quad (\alpha_i, \beta_j \in F). \quad (8)$$

*Proof.* Suppose  $\text{sp } A = \text{sp } PA$  for all  $n \times n$  permutation matrices  $P$ . Since the trace of a matrix equals the sum of its characteristic roots,

the trace of  $PA$  equals the trace of  $A$  for all  $P$ . It then follows, for every  $2 \times 2$  submatrix of  $A$

$$\begin{bmatrix} a_{ij} & a_{ik} \\ a_{lj} & a_{lk} \end{bmatrix} \quad (1 \leq i < l \leq n, 1 \leq j < k \leq n),$$

that  $a_{ij} + a_{lk} = a_{ik} + a_{lj}$  or

$$a_{ij} - a_{ik} = a_{lj} - a_{lk} \quad (1 \leq i < l \leq n, 1 \leq j < k \leq n).$$

This implies that the matrix  $A$  has the form (8), and, in particular, has rank at most 2. For  $X$  an  $n \times n$  matrix, let  $e_2(X)$  denote the second elementary symmetric function of the characteristic roots of  $X$ . Since for  $A$  as in (8) the trace of  $A$  equals the trace of  $PA$  for all  $n \times n$  permutation matrices  $P$ , and since at least  $n - 2$  characteristic roots of  $A$  are zero, it follows that  $\text{sp } A = \text{sp } PA$  for all  $P$  if and only if  $A$  is as in (8) with  $e_2(A) = e_2(PA)$  for all  $n \times n$  permutation matrices  $P$ . From (8), we conclude that

$$\begin{aligned} e_2(A) &= \sum_{n \geq i > j \geq 1} [(\alpha_i + \beta_i)(\alpha_j + \beta_j) - (\alpha_i + \beta_j)(\alpha_j + \beta_i)] \\ &= \sum_{n \geq i > j \geq 1} (\alpha_i - \alpha_j)(\beta_j - \beta_i). \end{aligned}$$

Let  $P_{k,l}$  denote the  $n \times n$  permutation matrix corresponding to the transposition  $(k, l)$  ( $1 \leq k < l \leq n$ ). A straightforward calculation shows that

$$e_2(A) - e_2(P_{k,l}A) = n(\alpha_k - \alpha_l)(\beta_l - \beta_k). \quad (9)$$

If the characteristic of  $F$  is zero or relatively prime to  $n$ , then from  $e_2(A) = e_2(PA)$  for all  $n \times n$  permutation matrices  $P$  we conclude by (9) that  $\alpha_k = \alpha_l$  or  $\beta_k = \beta_l$  ( $1 \leq k < l \leq n$ ). If  $\alpha_1 = \alpha_2 = \cdots = \alpha_n$ , then  $A$  is of the form (7). Otherwise the  $\alpha$ 's may be partitioned into two nonempty subsets  $S$  and  $T$  such that  $\alpha_i \in S, \alpha_j \in T$  imply  $\alpha_i \neq \alpha_j$ . This then implies  $\beta_1 = \beta_2 = \cdots = \beta_n$  and  $A^T$  is of the form (7).

If the characteristic of  $F$  divides  $n$ , then from (9) we conclude that  $e_2(A) = e_2(P_{k,l}A)$  ( $1 \leq k < l \leq n$ ). Moreover if  $A$  has the form (8) then so does  $QA$ , for every permutation matrix  $Q$ ; hence

$$e_2(QA) = e_2(P_{kl}QA). \quad (10)$$

Now let  $P$  be any  $n \times n$  permutation matrix. Then  $P$  has a representation of the form

$$P = P_{k_t, l_t} \cdots P_{k_2, l_2} P_{k_1, l_1}.$$

Hence from (10) we conclude

$$e_2(A) = e_2(P_{k_1, l_1} A) = e_2(P_{k_2, l_2} P_{k_1, l_1} A) = \cdots = e_2(PA).$$

Hence, in case the characteristic of  $F$  divides  $n$ , all matrices of the form (8) have the desired property. This completes the proof of Theorem 2.

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